A. Appendix
Appendix

A.1. Basic facts

It is convenient to introduce some notation for symmetric and Hermitian matrices. A real matrix A is symmetric if $A = A^T$. The sets of all $n \times n$ symmetric matrices will be denoted by $\mathbb{S}^n$. A complex matrix is Hermitian if $A = A^* = \overline{A}^T$ where the bar denotes the complex conjugate of each entry in A. The sets of all $n \times n$ Hermitian matrices will be denoted by $\mathbb{H}^n$.

Definition A.1. *Let A be an arbitrary matrix. $A_\perp$ denotes a matrix with the following properties.*

$$\mathbf{Ker}\,(A_\perp) = \mathbf{Im}\,(A) \quad \textit{and} \quad A_\perp A_\perp^* > 0, \tag{A.1}$$

or with other words $A_\perp^*$ is an arbitrary basis matrix in $\mathbf{Ker}\,(A^*)$.

Note that $A_\perp$ exists if and only if A has linearly dependent rows. Also note that, for a given A is not unique, but throughout this paper, any choice is acceptable. And finally, it is obvious that $A_\perp A = 0$, this latter property justifies our notation.

Definition A.2. *Let A be an arbitrary matrix. $A_\dashv$ denotes arbitrary basis matrix in $\mathbf{Ker}\,(A)$. Note that $A_\dashv$ exists if and only if A has linearly dependent columns. It is obvious that $AA_\dashv = 0$ and that $A_\dashv^* A_\dashv > 0$.*

A.1.1. The Moore-Penrose Pseudo-inverse

Definition A.3. *The pseudo-inverse $A^\dagger$ of an $m \times n$ matrix A (whose entries can be real or complex numbers) is defined as the unique $n \times m$ matrix* *satisfying all of the following four criteria:*

$$AA^{\dagger}A = A \tag{A.2}$$

$$A^{\dagger}AA^{\dagger} = A^{\dagger} \tag{A.3}$$

$$(AA^{\dagger})^{*} = AA^{\dagger} \tag{A.4}$$

$$(A^{\dagger}A)^{*} = A^{\dagger}A \tag{A.5}$$

Properties:

$A^{\dagger}$ exists and is unique for any matrix A. (A.6)

If A is invertible then $A^{\dagger} = A^{-1}$. (A.7)

$A^{\dagger}$ of a zero matrix is its transpose. (A.8)

$(A^{\dagger})^{\dagger} = A$. (A.9)

$(\alpha A)^{\dagger} = \alpha^{-1}A^{\dagger} \quad \text{for} \quad \alpha \neq 0$. (A.10)

$AA^{\dagger}$ orthogonal projector onto $\mathbf{Im}(A)$. (A.11)

$A^{\dagger}A$ orthogonal projector onto $\mathbf{Im}(A^{*})$. (A.12)

$(I - A^{\dagger}A)$ orthogonal projector onto $\mathbf{Ker}(A)$. (A.13)

$ker(A^{\dagger}) = (\mathbf{Im}(A))^{\perp}$. (A.14)

$im(A^{\dagger}) = (\mathbf{Ker}(A))^{\perp}$. (A.15)

If the columns of A are linearly independent, then $A^{*}A$ is invertible and:

$$A^{\dagger} = (A^{*}A)^{-1}A^{*} \quad \text{case} \quad m > n.$$

It follows that $A^{\dagger}$ is a left inverse of A, i.e., $A^{\dagger}A = I$.

If the rows of A are linearly independent, then AA^{*} is invertible and:

$$A^{\dagger} = A^{*}(AA^{*})^{-1} \quad \text{case} \quad m < n.$$

It follows that $A^{\dagger}$ is a right inverse of A, i.e., $AA^{\dagger} = I$.

A.1.2. Singular value decomposition

Theorem A.4 (SVD). *Suppose A is an $m \times n$ matrix whose entries come from the field $\mathbb{F}$, which is either the field of real numbers or the field of complex numbers. Then there exists a factorization of the form*

$$A = U\Sigma V^{*},$$

where U is an $m \times m$ unitary matrix over $\mathbb{F}$, the matrix Σ is $m \times n$ diagonal matrix with nonnegative real numbers on the diagonal, and V^* is an $n \times n$ unitary matrix over $\mathbb{F}$. Such a factorization is called a singular-value decomposition of A. A common convention is to order the diagonal entries $\Sigma_{i,i}$ in non-increasing fashion. In this case, the diagonal matrix Σ is uniquely determined by A (though the matrices U and V are not). The diagonal entries of Σ are known as the singular values of A. More precisely Σ has the form

$$\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_p)$$

where $p = min(m,n)$ and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0$$

Remark A.5 (Pseudo-inverse). *The singular value decomposition can be used for computing the pseudo-inverse of a matrix. Indeed, the pseudo-inverse of the matrix A with singular value decomposition $A = U\Sigma V^*$ is*

$$A^\dagger = V\Sigma^\dagger U^*,$$

where $\Sigma^\dagger$ is the pseudo-inverse of Σ with every nonzero entry replaced by its reciprocal.

A.1.3. Schur complement and Schur lemma

Lemma A.6 (Schur Decomposition). *Suppose A or D respectively is non non-singular. Then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$$

or

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}$$

Lemma A.7 (Schur Lemma). *Let Q and R be symmetric matrices. The following are equivalent.*

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0, \tag{A.16}$$

$$R \geq 0, \quad Q - SR^\dagger S^T \geq 0, \quad S(I - RR^\dagger) = 0 \tag{A.17}$$

$$Q \geq 0, \quad R - S^T Q^\dagger S \geq 0, \quad (I - QQ^\dagger)S = 0 \tag{A.18}$$

Lemma A.8 (Symmetric Schur Lemma). *Let Q and R be symmetric matrices. The following are equivalent.*

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} > 0, \tag{A.19}$$

$$R > 0, \quad Q - SR^{-1}S^T > 0. \tag{A.20}$$

$$Q > 0, \quad R - S^TQ^{-1}S > 0. \tag{A.21}$$

We note that the equality $S(I - RR^\dagger) = 0$ *is redundant since* $R^\dagger = R^{-1}$.

Lemma A.9. *Suppose that* $I - AB$ *is nonsingular. Then*

$$A(I - BA)^{-1} = (I - AB)^{-1}A$$

Lemma A.10 (Matrix Inversion Lemma). *Let A, C and* $D^{-1} + CA^{-1}B$ *be non-singular. Then*

$$(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}.$$

Suppose A and D are both non-singular. Then

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}.$$

Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Let us suppose that M is non-singular. Moreover, suppose A or D respectively is non-singular and let $V = D - CA^{-1}B$ *or* $W = A - BD^{-1}C$. *Then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BV^{-1}CA^{-1} & -A^{-1}BV^{-1} \\ -V^{-1}CA^{-1} & V^{-1} \end{pmatrix}$$

or

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} W^{-1} & -W^{-1}BD^{-1} \\ -D^{-1}CW^{-1} & D^{-1} + D^{-1}CW^{-1}BD^{-1} \end{pmatrix}$$

If M, A and D are all non-singular then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

A.1.4. Inertia

Definition A.11. *The inertia of a Hermitian $P \in \mathbb{R}^{n\times n}$ is defined as*

$$in(P) = (in_-(P), in_0(P), in_+(P)) \tag{A.22}$$

with $in_-(P)$, $in_0(P)$, $in_+(P)$ denoting the number of negative, zero and positive eigenvalues of P. Moreover, for any subspace $\mathcal{S} \subset \mathbb{R}^n$ the inertia $in(P|_\mathcal{S})$ is defined by $in(S^*PS)$, where S is an arbitrary basis matrix of $\mathcal{S}$.

Example A.12. *Consider $P = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$ and a (maximal) negative subspace $\mathcal{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. However, its complementary subspace $\mathcal{S}^\perp = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is also a (maximal) negative subspace! As it is expected $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ is not a negative subspace, since the eigenvalues of $\begin{pmatrix} -1+1/4 & 2+1/8 \\ 2+1/8 & -4+1/16 \end{pmatrix}$ are of different sign.*

But $\mathcal{S}^\perp$ is a positive subspace of P^{-1}.

Lemma A.13 (Inertia Lemma). *If A is nonsingular then*

$$in\begin{pmatrix} A & C \\ C^* & B \end{pmatrix} = in(A) + in(B - C^*A^{-1}C). \tag{A.23}$$

Lemma A.14 (Dualization Lemma). *Let P be a non-singular symmetric matrix in $\mathbb{R}^{n\times n}$ and let $\mathcal{U}$ and $\mathcal{V}$ be two complementary subspaces with $\mathcal{U} \oplus \mathcal{V} = \mathbb{R}^n$. Then*

$$x^T Px < 0 \textit{ for all } x \in \mathcal{U}\backslash\{0\} \textit{ and } x^T Px \geq 0 \textit{ for all } x \in \mathcal{V} \tag{A.24}$$

is equivalent to

$$x^T P^{-1} x > 0 \textit{ for all } x \in \mathcal{U}^\perp\backslash\{0\} \textit{ and } x^T P^{-1} x \leq 0 \textit{ for all } x \in \mathcal{V}^\perp. \tag{A.25}$$

An other formulation: let $\mathcal{S}$ be a subspace with $in_0(P|_\mathcal{S}) = 0$. Then

$$in(P) = in(P|_\mathcal{S}) + in(P^{-1}|_{\mathcal{S}^\perp}). \tag{A.26}$$

Example A.15. *Consider* $P = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{16} \end{pmatrix}$ *and a (maximal) negative subspace* $\mathcal{S} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *However, its complementary subspace* $\mathcal{S}^{\perp} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ *is also a (maximal) negative subspace! As it is expected* $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ *is not a negative subspace, since the eigenvalues of* $\begin{pmatrix} -1+1/4 & 2+1/8 \\ 2+1/8 & -4+1/16 \end{pmatrix}$ *are of different sign.*

But $\mathcal{S}^{\perp}$ *is a positive subspace of* P^{-1}.

A.2. Extensions of positive definite matrices

Lemma A.16. *Let the matrix P be partitioned as*

$$P = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \tag{A.27}$$

where $A > 0$ *and* $B > 0$. *Then* $P > 0$ *if and only if* $X = A^{1/2}KB^{1/2}$, *where* $\|K\| < 1$.

Proof:. The matrix P is similar to the matrix

$$\begin{pmatrix} A^{-1/2} & 0 \\ 0 & B^{-1/2} \end{pmatrix}\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\begin{pmatrix} A^{-1/2} & 0 \\ 0 & B^{-1/2} \end{pmatrix} = \begin{pmatrix} I & Y \\ Y^* & I \end{pmatrix} \tag{A.28}$$

with $Y = A^{-1/2}XB^{-1/2}$. Inequality

$$\begin{pmatrix} I & Y \\ Y^* & I \end{pmatrix} > 0 \tag{A.29}$$

is equivalent to $I - YY^* > 0$ and $I - Y^*Y > 0$ which means that $\|Y\| < 1$.

Consequently $P > 0$ if and only if $\|B^{-1/2}X^*A^{-1/2}\| < 1$, i.e., $X = A^{1/2}KB^{1/2}$, where K is an arbitrary contraction ($\|K\| < 1$).

Lemma A.17. *Let the matrix P be partitioned as*

$$P = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}. \tag{A.30}$$

If $A > 0$ *is given,* X *is arbitrary,* $B > 0$ *and* $B > X^*A^{-1}X$ *then* P *is positive definite.*

Proof:. Using the Schur complement, we have that $A > 0$, $B > 0$ and $B - X^* A^{-1} X > 0$ which implies that $P > 0$.

Theorem A.18. *[Positive definite extension] Let $X > 0$ and $Y > 0$ be given. Then there exists a positive definite matrix P that satisfy*

$$P = \begin{pmatrix} X & M \\ M^* & \overline{X} \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} Y & N \\ N^* & \overline{Y} \end{pmatrix}, \tag{A.31}$$

if and only is

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0. \tag{A.32}$$

If the existence condition is satisfied, all such extensions are given as

$$P = \begin{pmatrix} X^{1/2} & 0 \\ 0 & \overline{X}^{1/2} \end{pmatrix} \begin{pmatrix} I & K \\ K^* & I \end{pmatrix} \begin{pmatrix} X^{1/2} & 0 \\ 0 & \overline{X}^{1/2} \end{pmatrix} \tag{A.33}$$

with an arbitrary $\overline{X} > 0$ and a contraction K determined by the condition

$$KK^* = I - X^{-1/2} Y^{-1} X^{-1/2}. \tag{A.34}$$

Hence, the dimension of the minimal extension is given by

$$n_{X,Y} = \operatorname{rank}(X - Y^{-1}) = \operatorname{rank} \begin{pmatrix} X & I \\ I & Y \end{pmatrix}. \tag{A.35}$$

Proof:. The assertion is a direct consequence of the matrix inversion lemma. With the notation $W = X - M\overline{X}^{-1} M^*$ we have that $Y = W^{-1}$ and

$$\begin{pmatrix} X & M \\ M^* & \overline{X} \end{pmatrix}^{-1} = \begin{pmatrix} W^{-1} & -W^{-1} M \overline{X}^{-1} \\ -\overline{X}^{-1} M^* W^{-1} & \overline{X}^{-1} + \overline{X}^{-1} M^* W^{-1} M \overline{X}^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} W^{-1} & 0 \\ 0 & \overline{X}^{-1} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & \overline{X}^{-1} M^* \end{pmatrix} \begin{pmatrix} 0 & -W^{-1} \\ -W^{-1} & W^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & M \overline{X}^{-1} \end{pmatrix}. \tag{A.36}$$

From Lemma A.16 follows the equality $M = X^{1/2}K\overline{X}^{1/2}$. The expressions for M, N and $\overline{Y}$ can be obtained by direct computation:

$$M = X^{1/2}K\overline{X}^{1/2} \tag{A.37}$$

$$N = -YX^{1/2}K\overline{X}^{-1/2} = -YM\overline{X}^{-1} = -X^{-1/2}(I-KK^*)^{-1}K\overline{X}^{-1/2}, \tag{A.38}$$

$$Y = (X - M\overline{X}^{-1}M^*)^{-1} = X^{-1/2}(I-KK^*)^{-1}X^{-1/2}, \tag{A.39}$$

$$\overline{Y} = \overline{X}^{-1/2}\left[I + K^*(I-KK^*)^{-1}K\right]\overline{X}^{-1/2}. \tag{A.40}$$

The matrix P^{-1} can be expressed as

$$P^{-1} = W\begin{pmatrix} (I-KK^*)^{-1} & -(I-KK^*)^{-1}K \\ -K^*(I-KK^*)^{-1} & I+K^*(I-KK^*)^{-1}K \end{pmatrix}W \tag{A.41}$$

with

$$W = \begin{pmatrix} X^{-1/2} & 0 \\ 0 & \overline{X}^{-1/2} \end{pmatrix}, \tag{A.42}$$

or as

$$P^{-1} = \begin{pmatrix} Y & 0 \\ 0 & \overline{X}^{-1} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & \overline{X}^{-1}M^* \end{pmatrix}\begin{pmatrix} 0 & -Y \\ -Y & Y \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & MX^{-1} \end{pmatrix}. \tag{A.43}$$

Remark A.19. *If one has strict inequality in* (A.32) *it follows that we have* $I = X^{1/2}LY^{1/2}$ *with a suitable contraction* $\|L\| < 1$, *i.e.,* $X^{-1/2} = LY^{1/2}$.

Then (A.39) *reveals that* $I - KK^* = LL^*$.

A.3. Variable elimination

Lemma A.20 (Finsler's Lemma). *Let* $x \in \mathbb{R}^n$, $P = P^* \in \mathbb{R}^{n\times n}$ *and* $V \in \mathbb{R}^{q\times n}$ *with* $r = rank(V) < n$. *Then the following are equivalent*

$$\text{(1)} \quad P < 0 \quad on \quad \mathbf{Ker}(V) \tag{A.44}$$

$$\text{(2)} \quad (V_\perp)^T P V_\perp < 0 \quad on \quad \mathbb{R}^{n-r} \tag{A.45}$$

$$\text{(3)} \quad \exists \mu \in \mathbb{R} : P - \mu V^T V < 0 \quad on \quad \mathbb{R}^n \tag{A.46}$$

$$\text{(4)} \quad \exists X \in \mathbb{R}^{n\times q} : P + XV + V^T X^T < 0 \quad on \quad \mathbb{R}^n \tag{A.47}$$

Inequality (3) *can be replaced by the variant*

$$\textit{(3b)} \quad \exists X = X^T \in \mathbb{R}^{q\times q} : P + V^T XV < 0 \quad \textit{on} \quad \mathbb{R}^n. \tag{A.48}$$

Remark A.21. *Inequality* (4) *can be also written in the form*

$$P + \begin{pmatrix} I \\ V \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix} < 0, \tag{A.49}$$

and also in the form

$$\begin{pmatrix} I \\ XV \end{pmatrix}^T \begin{pmatrix} P & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I \\ XV \end{pmatrix} < 0. \tag{A.50}$$

Lemma A.22 (Projection Elmma). *For arbitrary A, B and a symmetric P, the LMI*

$$P + AXB + (AXB)^* < 0 \tag{A.51}$$

in the unstructured X has a solution if and only if

$$A^* x = 0 \quad \textit{or} \quad Bx = 0 \quad \textit{imply} \quad x^T Px < 0 \quad \textit{or} \quad x = 0. \tag{A.52}$$

The conditions above are equivalent to

$$A_\perp PA_\perp^* < 0 \quad \textit{and} \quad B_\dashv^* PB_\dashv < 0. \tag{A.53}$$

Remark A.23. *Inequality* (A.51) *can be also written in the form*

$$P + \begin{pmatrix} A^T \\ B \end{pmatrix}^T \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \begin{pmatrix} A^T \\ B \end{pmatrix} < 0, \tag{A.54}$$

and also in the form

$$\begin{pmatrix} I \\ AXB \end{pmatrix}^T \begin{pmatrix} P & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I \\ AXB \end{pmatrix} < 0. \tag{A.55}$$

Inequalities in (A.53) *can also be formulated as $(A^*)_\dashv^* P(A^*)_\dashv < 0$ and $(B^*)_\perp P(B^*)_\perp^* < 0$, respectively.*

Lemma A.24 (Elimination Lemma). *Let $Q = Q^T$ non-singular with in(Q) = $(m,0,n)$ and let us consider the quadratic matrix inequality*

$$\begin{pmatrix} I \\ C+AXB \end{pmatrix}^T Q \begin{pmatrix} I \\ C+AXB \end{pmatrix} < 0. \tag{A.56}$$

Here C is of dimension $n \times m$. This inequality has a solution if and only if

$$B_{\dashv}^T \begin{pmatrix} I \\ C \end{pmatrix}^T Q \begin{pmatrix} I \\ C \end{pmatrix} B_{\dashv} < 0 \tag{A.57}$$

and

$$A_{\perp} \begin{pmatrix} -C^T \\ I \end{pmatrix}^T Q^{-1} \begin{pmatrix} -C^T \\ I \end{pmatrix} A_{\perp}^T > 0. \tag{A.58}$$

A.4. The Möbius transformation

Definition A.25. *Let $M \in \mathbb{F}^{(m+n)\times(m+n)}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) be partitioned as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{A.59}$$

The Möbius transformation T_M is defined by the equation

$$T_M(X) = (C+DX)(A+BX)^{-1} \tag{A.60}$$

for $X \in \mathbb{F}^{n\times m}$ where $(A+BX)^{-1}$ exists. Denote by

$$\operatorname{dom}(T_M) = \left\{ X \in \mathbb{F}^{n\times m} \;:\; \exists (A+BX)^{-1} \right\} \tag{A.61}$$

the domain of T_M.

The dual Möbius transformation is defined by

$$T_M^d(Z) = (ZB+D)^{-1}(ZA+C), \tag{A.62}$$

and

$$\operatorname{dom}(T_M^d) = \left\{ Z \in \mathbb{F}^{n\times m} \;:\; \exists (ZB+D)^{-1} \right\}. \tag{A.63}$$

Theorem A.26. *Let $M \in \mathbb{F}^{(m+n)\times(m+n)}$. Then*

$$X \in \operatorname{dom}(T_M^d) \quad \Leftrightarrow \quad X^* \in \operatorname{dom}(T_{L^*M^*L}). \tag{A.64}$$

Moreover

$$T_M^d(X) = T^*_{L^*M^*L}(X^*), \tag{A.65}$$

where

$$L = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}. \tag{A.66}$$

If $M \in \mathbb{F}^{(m+n)\times(m+n)}$ *is a nonsingular matrix, then*

$$T_M(X) = -T^d_{M^{-1}}(-X). \tag{A.67}$$

Proof:. A direct computation reveals that

$$\left[T_M^d(X)\right]^* = (C^* + A^*X^*)(D^* + B^*X^*)^{-1} = T_{L^*M^*L}(X^*). \tag{A.68}$$

Let M and M^{-1} be partitioned as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \tag{A.69}$$

then identity

$$\begin{pmatrix} -X & I_n \end{pmatrix} M^{-1} M \begin{pmatrix} I_m \\ X \end{pmatrix} = 0 \tag{A.70}$$

implies that

$$(H - XF)\begin{pmatrix} T^d_{M^{-1}}(-X) & I_n \end{pmatrix}\begin{pmatrix} I_m \\ T_M(X) \end{pmatrix}(A + BX) = 0, \tag{A.71}$$

provided $(H - XF)$ and $(A + BX)$ are nonsingular. Then

$$\begin{pmatrix} T^d_{M^{-1}}(-X) & I_n \end{pmatrix}\begin{pmatrix} I_m \\ T_M(X) \end{pmatrix} = 0, \tag{A.72}$$

i.e., (A.67).

It remains to prove that $X \in \mathrm{dom}(T_M)$ is equivalent to $-X \in \mathrm{dom}(T^d_{M^{-1}})$. To this end consider the nonsingular matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} A + BX & -AX^* + B \\ C + DX & D - CX^* \end{pmatrix} \tag{A.73}$$

and its inverse

$$T^{-1} = \begin{pmatrix} I & -X^* \\ X & I \end{pmatrix}^{-1} \begin{pmatrix} E & F \\ G & H \end{pmatrix} =$$
$$= \begin{pmatrix} (I+X^*X)^{-1} & X^*(I+XX^*)^{-1} \\ -(I+XX^*)^{-1}X & (I+XX^*)^{-1} \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}. \qquad \text{(A.74)}$$

If $(A+BX)$ is nonsingular then from the Schur inversion formula it follows that the right bottom block of T^{-1} is also nonsingular. This block equals to $(I+XX^*)^{-1}(H-XF)$, hence $(H-XF)$ is nonsingular. Analogously, nonsingularity of $(H-XF)$ implies the nonsingularity of $A+BX$.

Corollary A.27.

$$-T_M^*(X) = T_{L^*M^{-*}L}(-X^*). \qquad \text{(A.75)}$$

Let us consider the composition of two Möbius transformations.

Definition A.28. *Let M and N be matrices partitioned as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad N = \begin{pmatrix} E & F \\ G & H \end{pmatrix}. \qquad \text{(A.76)}$$

Then the composition of the transformations T_M and T_N is defined by

$$(T_N \circ T_M)(X) = T_N(T_M(X)). \qquad \text{(A.77)}$$

Lemma A.29.

$$(T_N \circ T_M)(X) = T_N(T_M(X)) = T_{NM}(X), \qquad \text{(A.78)}$$

$$X \in \mathrm{dom}(T_M) \quad \textit{and} \quad T_M(X) \in \mathrm{dom}(T_N) \qquad \text{(A.79)}$$

or equivalently

$$X \in \mathrm{dom}(T_M) \quad \textit{and} \quad X \in \mathrm{dom}(T_{NM}). \qquad \text{(A.80)}$$

If M is nonsingular, $X \in \mathrm{dom}(T_M)$ and $T_M(X) = K$ then $K \in \mathrm{dom}(T_{M^{-1}})$ and $T_{M^{-1}}(K) = X$, i.e.,

$$\mathrm{dom}(T_M) = \mathrm{Range}(T_{M^{-1}}). \qquad \text{(A.81)}$$

A.5. Kalman-Yakubovich-Popov type results

Lemma A.30 (Robust Finsler's lemma). *Let fixed matrices $Q = Q^*$, W and a compact subset of matrices $\mathcal{H}$ be given.*

Then the following statements are equivalent:

i.) for each $H \in \mathcal{H}$

$$\xi^* Q\xi < 0, \quad \forall\xi \neq 0, \ HW\xi = 0. \tag{A.82}$$

ii.) there exists $\Theta = \Theta^*$ such that

$$Q + W^*\Theta W < 0, \tag{A.83}$$

$$\mathcal{N}_H^*\Theta\mathcal{N}_H \geq 0, \quad \forall H \in \mathcal{H}. \tag{A.84}$$

This result is a generalization of the Finsler's lemma. A similar, slightly more general, result is called the full block S-procedure.

Lemma A.31. *Let matrices A, B, C and Q be given, where all the matrices except B are symmetric. Then the following statements are equivalent.*

$$(i) \text{There is an } X \text{ such that } \begin{pmatrix} Q & X \\ X^* & R \end{pmatrix} > \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{A.85}$$

$$(ii) \quad F = Q - A > 0 \quad \text{and} \quad G = R - C > 0. \tag{A.86}$$

If the above statements hold all X are given as

$$X = B + F^{1/2}LG^{1/2}, \tag{A.87}$$

where L is an arbitrary contraction such that $\|L\| < 1$.

Proof:. The proof is elementary, hence it is omitted for brevity.

A.5.1. Variants of the KYP lemma

Theorem A.32 (Extended KYP lemma). *Let P be a Hermitian matrix. Then*

$$\begin{pmatrix} F(\delta) \\ I \end{pmatrix}^* P \begin{pmatrix} F(\delta) \\ I \end{pmatrix} \prec 0, \quad \forall\delta \in \mathbf{\Delta} \tag{A.88}$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$, if there exists a Hermitian multiplier Q which satisfies

C-1*:*

$$\begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^* Q \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}^* P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \prec 0,$$

C-2*:*

$$\begin{pmatrix} I \\ \delta \end{pmatrix}^* Q \begin{pmatrix} I \\ \delta \end{pmatrix} \geq 0, \quad \forall \delta \in \mathbf{\Delta}.$$

For compact $\mathbf{\Delta}$ *one has the reverse implication, too.*

Proof:. Recall the fact that $\delta(I - A\delta)^{-1} = (I - \delta A)^{-1}\delta$ and consider

$$G(\delta) = \begin{pmatrix} (I - \delta A)^{-1}\delta B \\ I \end{pmatrix}. \tag{A.89}$$

Then one has the identities

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} G(\delta) = \begin{pmatrix} \delta \\ I \end{pmatrix} (I - A\delta)^{-1} B \tag{A.90}$$

and

$$\begin{pmatrix} C & D \\ 0 & I \end{pmatrix} G(\delta) = \begin{pmatrix} F(\delta) \\ I \end{pmatrix}. \tag{A.91}$$

Therefore from **C-1** one has

$$((I - A\delta)^{-1}B)^* \begin{pmatrix} \delta \\ I \end{pmatrix}^* Q \begin{pmatrix} \delta \\ I \end{pmatrix} (I - A\delta)^{-1} B + \begin{pmatrix} F(\delta) \\ I \end{pmatrix}^* P \begin{pmatrix} F(\delta) \\ I \end{pmatrix} \prec 0. \tag{A.92}$$

From **C-1** it follows (A.88).

For the reverse implication let us consider that $\mathbf{\Delta}$ is compact.

Observe that $G_\perp(\delta) = \begin{pmatrix} (I - \delta A) & -\delta B \end{pmatrix}$, i.e.,

$$G_\perp(\delta) G(\delta) = \begin{pmatrix} -\delta & I \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} \delta(I - A\delta)^{-1} B \\ I \end{pmatrix} = 0. \tag{A.93}$$

Then by an application of the robust Finsler's lemma, i.e., by taking $H = \begin{pmatrix} -\delta & I \end{pmatrix}$ and $W = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}$ the assertion follows.

Theorem A.33 (Extended KYP with transformed parameters). *For a given compact set* $\tilde{\mathbf{\Delta}}$ *let us consider a nonsingular matrix* M*, the corresponding Möbius transformation* T_M *and a set* $\mathbf{\Delta}$ *such that* $\mathbf{\Delta} \subset \mathrm{dom}(T_M)$ *and* $\tilde{\mathbf{\Delta}} = T_M(\mathbf{\Delta})$.

Then

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} \prec 0, \quad \forall \delta \in \mathbf{\Delta} \tag{A.94}$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$*, if and only if there exists a symmetric (Hermitian) multiplier* P *which satisfies*

M-1*:*

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0, \tag{A.95}$$

M-2*:*

$$P = \overline{M}^* \tilde{P} \overline{M}, \quad \text{with} \begin{pmatrix} \tilde{\delta} \\ I \end{pmatrix}^* \tilde{P} \begin{pmatrix} \tilde{\delta} \\ I \end{pmatrix} \geq 0, \quad \forall \tilde{\delta} \in \tilde{\mathbf{\Delta}}, \tag{A.96}$$

where $\overline{M} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} M \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Proof:. Recall the fact that $\delta(I - A\delta)^{-1} = (I - \delta A)^{-1}\delta$ and consider

$$G(\delta) = \begin{pmatrix} (I - \delta A)^{-1}\delta B \\ I \end{pmatrix}. \tag{A.97}$$

Then one has the identities

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} G(\delta) = \begin{pmatrix} \delta \\ I \end{pmatrix} (I - A\delta)^{-1} B \tag{A.98}$$

and

$$\begin{pmatrix} 0 & I \\ C & D \end{pmatrix} G(\delta) = \begin{pmatrix} I \\ F(\delta) \end{pmatrix}. \tag{A.99}$$

If M is partitioned as $M = \begin{pmatrix} U & V \\ X & Z \end{pmatrix}$ by using the notation $W(\delta) = (U + V\delta)^{-1}$ one has

$$\begin{pmatrix} \tilde{\delta} \\ I \end{pmatrix}^* \tilde{P} \begin{pmatrix} \tilde{\delta} \\ I \end{pmatrix} = W(\delta)^* \begin{pmatrix} \delta \\ I \end{pmatrix}^* \overline{M}^* \tilde{P} \overline{M} \begin{pmatrix} \delta \\ I \end{pmatrix} W(\delta). \tag{A.100}$$

It follows that with $P = \overline{M}^* \tilde{P} \overline{M}$ one has

$$\begin{pmatrix}\tilde{\delta}\\ I\end{pmatrix}^* \tilde{P} \begin{pmatrix}\tilde{\delta}\\ I\end{pmatrix} \geq 0 \text{ iff } \begin{pmatrix}\delta\\ I\end{pmatrix}^* P \begin{pmatrix}\delta\\ I\end{pmatrix} \geq 0. \tag{A.101}$$

Therefore, from **M-1** one has

$$((I - A\delta)^{-1}B)^* \begin{pmatrix}\delta\\ I\end{pmatrix}^* P \begin{pmatrix}\delta\\ I\end{pmatrix} (I - A\delta)^{-1}B + \tag{A.102}$$

$$+ \begin{pmatrix}I\\ F(\delta)\end{pmatrix}^* P_p \begin{pmatrix}I\\ F(\delta)\end{pmatrix} \prec 0. \tag{A.103}$$

Thus, from **M-1** and **M-2** follows (A.94).

For the reverse implication observe that

$$G_\perp(\delta) = \begin{pmatrix}(I - \delta A) & -\delta B\end{pmatrix}, \tag{A.104}$$

i.e., from (A.90) one has

$$G_\perp(\delta)G(\delta) = \begin{pmatrix}I & -\delta\end{pmatrix} \begin{pmatrix}I & 0\\ A & B\end{pmatrix} G(\delta) = 0. \tag{A.105}$$

Using Theorem A.26 it follows that

$$-\delta = -T_{M^{-1}}(\tilde{\delta}) = T_M^d(-\tilde{\delta}). \tag{A.106}$$

Thus, one has

$$\begin{pmatrix}-\delta & I\end{pmatrix} = \begin{pmatrix}T_M^d(-\tilde{\delta}) & I\end{pmatrix} = \tag{A.107}$$

$$= (-\tilde{\delta}V + Z)^{-1} \begin{pmatrix}-\tilde{\delta}U + X & -\tilde{\delta}V + Z\end{pmatrix} = \tag{A.108}$$

$$= (-\tilde{\delta}V + Z)^{-1} \begin{pmatrix}-\tilde{\delta} & I\end{pmatrix} M. \tag{A.109}$$

It follows that $G_\perp(\delta)\xi = 0$ is equivalent to

$$\begin{pmatrix}I & -\tilde{\delta}\end{pmatrix} \overline{M} \begin{pmatrix}I & 0\\ A & B\end{pmatrix} \xi = 0 \tag{A.110}$$

for all ξ.

Since $\tilde{\mathbf{\Delta}}$ is compact one can apply the robust Finsler's lemma, i.e., by taking $H = \begin{pmatrix} I & -\tilde{\delta} \end{pmatrix}$ and $W = \overline{M}\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}$ in Lemma A.30, one has that there exist a matrix $\tilde{P}$ such that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0, \tag{A.111}$$

with $P = \overline{M}^* \tilde{P} \overline{M}$ and

$$\begin{pmatrix} \tilde{\delta} \\ I \end{pmatrix}^* \tilde{P} \begin{pmatrix} \tilde{\delta} \\ I \end{pmatrix} \geq 0, \quad \forall \delta \in \tilde{\mathbf{\Delta}}. \tag{A.112}$$

Remark A.34. *In* **M-2** *one can always take strict inequality without restricting generality.*

Proposition A.33 extends the power of the S-procedure to sets that are not bounded, however, that can be obtained as a Möbis transform of a compact set. As an example one can obtain an easy derivation of the strict version of the Kalman-Yakubovich-Popov lemma:

Theorem A.35 (KYP Continuous Time). *Let the matrices* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$ *be given, with* $\det(\mathrm{j}\omega I - \mathrm{A}) \neq 0$ *for* $\omega \in \mathbb{R}$.

The following two statements are equivalent:

i.)

$$\begin{pmatrix} (\mathrm{j}\omega I - A)^{-1} B \\ I \end{pmatrix}^* M \begin{pmatrix} (\mathrm{j}\omega I - A)^{-1} B \\ I \end{pmatrix} < 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$.

ii.) There exists a matrix $P \in \mathbb{R}^{n\times n}$ *such that* $P = P^T$ *and*

$$M + \begin{pmatrix} A^T P + PA & PB \\ B^T P & 0 \end{pmatrix} < 0.$$

Theorem A.36 (KYP Discrete Time). *Let the matrices* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$ *be given, with* $\det(e^{\mathrm{j}\omega}\omega I - \mathrm{A}) \neq 0$ *for* $\omega \in \mathbb{R}$.

The following two statements are equivalent:

i.)

$$\begin{pmatrix} (e^{\mathrm{j}\omega}\omega I - A)^{-1} B \\ I \end{pmatrix}^* M \begin{pmatrix} (e^{\mathrm{j}\omega}\omega I - A)^{-1} B \\ I \end{pmatrix} < 0$$

for all $\omega \in \mathbb{R}$.

ii.) There exists a matrix $P \in \mathbb{R}^{n\times n}$ *such that* $P = P^T$ *and*

$$M + \begin{pmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB \end{pmatrix} < 0.$$

Remark A.37. *If* (A, B) *is controllable the corresponding equivalence also holds for non-strict inequalities.*

The discrete-time version of the lemma can be obtained by the compactness of the unit circle while the continuous-time version follows from the fact that the imaginary line is a Möbius transform (Cayley-transform) of the unit circle.